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Excited States in the Twisted XXZ Spin Chain

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Abstract

We compute the finite size spectrum for the spin 1/2 XXZ chain with twisted boundary conditions, for anisotropy in the regime $0 < \gamma < \pi/2$, and arbitrary twist θ . The string hypothesis is employed for treating complex excitations. The Bethe Ansatz equations are solved within a coupled non-linear integral equation approach, with one equation for each type of string. The root-of-unity quantum group invariant periodic chain reduces to the XXZ_{1/2} chain with a set of twist boundary conditions ($\pi/\gamma \in \mathbb{Z}$, θ an integer multiple of γ). For this model, the restricted Hilbert space corresponds to an unitary conformal field theory, and we recover all primary states in the Kač table in terms of states with specific twist and strings.

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1 Introduction

Apart from their relevance to real lattice systems with restricted dimensionality, spin chains are interesting from a field theoretical point of view as integrable lattice regularizations of 1+1 dimensional field theories [1].

The continuum limit of the field theory is the thermodynamic limit of the spin chain, where the number of spins in the chain goes to infinity. If the thermodynamic limit is accompanied by taking the lattice spacing to zero, keeping the length of the chain constant, we get a description of the continuum field theory in finite volume. At criticality, one expects conformal invariance, with the conformal weights read out via the finite size corrections of the energy and the momentum [2, 3, 4].

In the class of theories with trigonometric R -matrices, massive theories (sine-Gordon/massive Thirring models) are usually associated to spin-1/2 anisotropic Heisenberg chains (XXZ_{1/2} chains) with inhomogeneities [5], whereas massless theories are related to homogeneous chains. The conformal properties of the latter are well known, the central charge $c = 1$, and conformal weights are ones of the gaussian model [6, 7, 8].

Central charges $c < 1$ can be reached by considering chains with some special fixed, or twisted boundary conditions, related to critical Potts and Ashkin-Teller chains [6, 9, 10, 11]. The correct interpretation of this lowering of c in the case of open chains was found by Pasquier and Saleur [12]. The key feature is the invariance of the spin chain dynamics under the universal quantum enveloping algebra $U_q(sl_2)$. (The quantum deformation parameter is given by the spin chain anisotropy as $q = e^{i\gamma}$). For open chains, the boundary conditions can be chosen to ensure $U_q(sl_2)$ invariance. This, however, is not sufficient to alter the central charge, which is a bulk effect. The lowering of c comes about for a root-of-unity q (rational γ/π), when the quantum group invariance allows for a self-consistent truncation of the Hilbert space. By restricting the Hilbert space to type-II (a.k.a “good”) representations of the quantum group, the central charge is lowered from $c = 1$ to those of minimal models [12]. Furthermore, analyzing low lying excited states shows that the conformal weights of primary operators form a one-parameter subset of the Kač table.

For closed chains, quantum group invariance is a more delicate matter [13, 14]. A topological interaction of the Wess-Zumino-Witten type has to be introduced [13], which leads to non-local terms in the Hamiltonian. On the level of Bethe Ansatz equations, the quantum group invariant closed chain can be viewed as a collection of XXZ_{1/2} chains with a set of twisted boundary conditions commensurate with the anisotropy. Again, quantum group reduction leads to a lowering of the central charge to $c < 1$ [14, 15]. The collection of twist boundary conditions is the exact analogue to the θ -vacua of SOS and sine-Gordon models, in which the quantum group restriction leads to RSOS or RSG models, respectively. The quantum group invariant chain can be considered as the critical/ultraviolet limits of these models (see e.g. Ref. [16]).

The main motivation for this work is the interpretation found in Reference [17], of the conformal properties of twisted XXZ_{1/2} chains, in terms of a lagrangean conformally invariant field theory with $c < 1$, namely the Liouville model with imaginary coupling. If the imaginary Liouville model on a circle is discretized preserving the integrable structure, it is equivalent to the quantum group invariant

periodic $XXZ_{1/2}$ chain on the level of Bethe Ansatz; the Liouville model can be mapped to a set of $XXZ_{1/2}$ chains with twisted boundary conditions. For root of unity couplings, a quantum group reduction can be performed on the Hilbert space. The central charges of minimal models are reproduced, confirming the dependence of central charge on Liouville coupling, known from canonical quantization.

Furthermore, based on Karowski's treatment of Potts models [18], primary states furnishing the full Kač tables of unitary minimal models were conjectured to arise from a specific set of Bethe Ansatz states. More exactly, primary states should arise from BA states with one string excitation above the sea of real BA rapidities, and no extra holes, i.e. vanishing total spin S of the chain.

This leads us to reconsider the finite size analysis of $XXZ_{1/2}$ chains with twisted boundary conditions. In the extensive literature on the subject [19, 7, 8, 9, 10, 18, 20, 21, 22, 23, 24, 25, 15], the main stress has been on investigations of pure hole excitations. Very little has been done to analyze complex excitations. String excitations with a restricted set of couplings and twists were investigated in [18], and complex pair excitations were considered in [22]. Apart from these, most work on complex excitations has been numerical [9, 24, 25].

We shall use a non-linear integral equation (NLIE) method [26, 27] to analyze the finite size spectrum. This method is more powerful than the Euler-MacLaurin methods mostly used in the literature, and it allows for extracting the scaling behaviour from the Bethe Ansatz equations in a rather straight forward manner. We follow the Destri-de Vega (DdV) approach [27], extending it to complex excitations with a twist. In Reference [28], the DdV equations for the sine-Gordon model were generalized to complex excitations described as wide and close pairs. Here, we make the crucial departure from the methods of [28] in that the string picture of the excitations is used. This is because, as pointed out above, the primary states are most transparently described in this language.

The plan of the paper will be the following: In Section 2, we formulate the algebraic Bethe Ansatz for the $XXZ_{1/2}$ chain with twisted boundary conditions. This we do for generic anisotropy in the antiferromagnetic regime $0 < \gamma < \pi/2$ and generic twist angle θ . We describe the complex excitations as strings. The DdV equations are derived for the case where we have positive parity strings of one specified length k in addition to a bulk of real Bethe Ansatz roots. To facilitate the analysis, we restrict the range of string lengths to obey $k\gamma < \pi$. In Section 3, we compute the finite size corrections to the energies and momenta of these states. We specialize to states where no holes punctuate the distribution of real roots. In Section 4, we specialize to the Hilbert space pertinent for lattice Liouville model, i.e. the restricted quantum group invariant chain. We show how the quantum group reduction lowers the central charge. Analyzing excited states with only one longer string in the sector $S^z = 0$, we show that these states are primary states of the theory, furnishing the whole Kač table.

The main point of this paper will be the fact that taking Bethe Ansatz configurations with one positive parity string, we are able to retrieve the spectrum of unitary minimal conformal field theories using a NLIE method. The two integers in the ensuing Kač table are the string length k and the twist κ . Aside from the technical advantage gained from this way of doing things, the hope is that this method is easier to generalize to the less explored case of $c > 25$ of relevance to Liouville theory with real coupling.

2 Twisted Bethe Ansatz equations for strings

We consider a spin 1/2 XXZ chain with $2N$ sites, with anisotropy γ , and boundary conditions twisted by the angle θ :

$$\sigma_{2N+1}^{\pm} = e^{\pm i\theta} \sigma_1^{\pm} ; \quad \sigma_{2N+1}^z = \sigma_1^z . \quad (1)$$

The corresponding hamiltonian reads:

$$H_{XXZ} = \frac{1}{2} \sum_{j=1}^{2N} \sigma_j^+ \sigma_{j+1}^- + \sigma_j^- \sigma_{j+1}^+ + 2 \cos \gamma \sigma_j^z \sigma_{j+1}^z \quad (2)$$

where the antiferromagnetic XXX limit corresponds to $\gamma = 0$ and the free Fermion point is at $\gamma = \frac{\pi}{2}$.

We formulate the theory in the standard language of quantum inverse scattering [29]. The L-matrix, satisfying the fundamental commutation relation $RLL = LLR$, is

$$L_{n,a}(\lambda) = \frac{1}{\sinh i\gamma} \begin{pmatrix} \sinh(\lambda + i\gamma \sigma_n^z) & \sigma_n^- \sinh i\gamma \\ \sigma_n^+ \sinh i\gamma & \sinh(\lambda - i\gamma \sigma_n^z) \end{pmatrix} . \quad (3)$$

where the operators $\sigma_n^z, \sigma_n^{\pm}$ are defined by the commutation relations of the quantum group $U_q(sl_2)$. The L-matrix is normalized so that for $\lambda = i\gamma/2$ it degenerates to a permutation operator between the auxiliary and quantum spaces. Using this L-matrix, we can write the transfer matrix of the chain, and build the eigenstates of the transfer matrix using the algebraic Bethe Ansatz.

We will focus our attention on those eigenstates of the transfer matrix that can be described in terms of real roots λ_j , $j = 1, \dots, n_1$, as well as strings of complex roots $\lambda_j^{(l)}$, $j = 1, \dots, n_l$. A (positive parity) l -string $\lambda^{(l)}$ is the collection of l roots of the form $\lambda + in\gamma$, $n = -(l-1)/2, \dots, (l-1)/2$, with λ real. Accordingly, in compact notations real roots are frequently designated as one-strings.

We shall only consider positive parity strings. Negative parity strings can be treated within the same context but they are irrelevant for our purposes since they do not lead to primary conformal states in the thermodynamic limit.

The number of l -strings is n_l , and the total number of roots is $\sum_l l n_l = N - S$, where S is the number of Bethe roots lacking from the maximal number of roots N , i.e. the z -component of the total spin of the chain. In order to simplify the underlying analysis, we consider only the antiferromagnetic regime $\gamma < \pi/2$, which covers all unitary theories. We also adopt a technical restriction on the range of the string length, so that $k\gamma < \pi$. This restriction, although not required in the most general case of anisotropy, shows up rather naturally when we consider the roots of unity case of the XXZ model pertaining to unitary conformal field theories, as a consequence of the underlying quantum group structure.

The Bethe Ansatz equations solved by the rapidities $\lambda_i^{(k)}$ describing a Bethe state are

$$e^{-2ik\theta} \left[\frac{\sinh(\lambda_i^{(k)} + ik\gamma/2)}{\sinh(\lambda_i^{(k)} - ik\gamma/2)} \right]^{2N} = - \prod_{l, n_l \neq 0} \prod_{j=1}^{n_l} S_{kl}(\lambda_i^{(k)} - \lambda_j^{(l)}) , \quad \forall(k, i) , \quad (4)$$

where the string-string scattering matrix is

$$S_{kl}(\lambda) = \prod_{m=|k-l|/2}^{(k+l)/2-1} \frac{\sinh(\lambda + i m \gamma)}{\sinh(\lambda - i m \gamma)} \frac{\sinh(\lambda + i(m+1)\gamma)}{\sinh(\lambda - i(m+1)\gamma)} . \quad (5)$$

To solve these equations, it is beneficial to write them in a logarithmic form. For this, we define the phase function

$$\begin{aligned} \phi_m(\lambda) &= i \log \frac{\sinh(i m \gamma + \lambda)}{\sinh(i m \gamma - \lambda)}, \quad m > 0 \\ \phi_0(\lambda) &= 0 . \end{aligned} \quad (6)$$

For real argument λ , the phase function ϕ_m is a continuous monotonic function. We choose the branch of the logarithm so that it is antisymmetric with respect to the origin. Using the phase function, we further define the momentum function

$$p_l(\lambda) = \phi_{l/2}(\lambda) , \quad (7)$$

the string-string scattering phase

$$\Phi_{kl}(\lambda) = \sum_{m=|k-l|/2}^{(k+l)/2-1} [\phi_m(\lambda) + \phi_{m+1}(\lambda)] , \quad (8)$$

and the counting functions for l -strings

$$z_l(\lambda) = 2Np_l(\lambda) - \sum_{k,j} \Phi_{lk}(\lambda - \lambda_j^{(k)}) + 2l\theta . \quad (9)$$

By defining counting functions for strings, all the analysis is reduced to dealing with functions of real variables, a considerable simplification over the usual method of treating complex NLIE's. In this respect, the string hypothesis is necessary, (although probably not inescapable), and marks a departure from the standard Destri-de Vega treatment. Of course, we must keep in mind that this is only justified when we are at zero temperature.

In terms of the counting functions, the Bethe equations (4) become

$$z_l(\lambda_i^{(l)}) = 2\pi I_i^{(l)} , \quad (10)$$

where the quantum numbers $I_i^{(l)}$ encode the choice of branch of the logarithm. They are integers or half integers when n_l is even or odd, respectively. These quantum numbers should all be distinct for a specified string length, otherwise the Bethe Ansatz equations are easily seen to lead to a infinite repulsion of the quasiparticles corresponding to equal quantum numbers. This is what is usually referred to as the fermionic character of Bethe states.

The basic assumption one makes in treating the Bethe equations for real rapidities λ is that the monotonicity of $2Np_1$ is sufficient to make the counting function z_1 overall monotonic. This allows one to find the spectrum of the quantum numbers $I_i^{(1)}$. Indeed, assuming monotonicity, $z_1/2\pi$ takes all the (half) integer values between $z_1(-\infty)/2\pi$ and $z_1(\infty)/2\pi$ exactly once. The only freedom we are left with in determining the spectrum of the I 's lies in the choice of the overall branch of the counting function z_1 , which is irrelevant. Any possible non-monotonicity would

induce additional holes close to the ends of the rapidity distributions [28]. These don't effect the quantities calculated here. As opposed to z_1 , the higher string counting functions z_k are generically non-monotonous.

For a configuration characterized by the higher string occupation numbers n_k , the asymptotic values of the counting functions are

$$z_l(\pm\infty) = \pm\pi n_l \pm 2S(\pi - l\gamma) \pm 2\pi \sum_{k>l} (k-l)n_k + 2l\theta. \quad (11)$$

If $n_k \neq 0$ for some k , it is clear that there are more vacancies between $z_1(-\infty)$ and $z_1(\infty)$ than those occupied by the Bethe Ansatz roots. These unoccupied vacancies are called holes, and they correspond to zeros of the eigenvalue of the transfer matrix (as opposed to Bethe Ansatz roots, which correspond to poles with vanishing residue). The number of holes in the distribution of real roots is

$$h = 2 \sum_{k>1} (k-1)n_k + 2S + \left\lfloor \frac{1}{2} - \frac{\theta}{\pi} - S\frac{\gamma}{\pi} \right\rfloor - \left\lfloor \frac{1}{2} - \frac{\theta}{\pi} + S\frac{\gamma}{\pi} \right\rfloor. \quad (12)$$

Here $\lfloor \circ \rfloor$ denotes the integer part of \circ . The integer part terms vanish for small values of the total spin S . The salient feature of Relation (12) is that a k -string gives rise to $2(k-1)$ holes.

The antiferromagnetic ground state is given by the configuration where there are no complex roots, and the number of real roots is maximal, i.e. $n_1 = N$; $n_k = 0$, $k > 1$; $S = 0$. From Equation (12), it is easily seen that there are no holes; in the ground state all vacancies are filled.

In the thermodynamical limit, low-lying excitations above the ground state are characterized by configurations where some higher strings are excited, as well as possibly some extra holes (S might differ from zero), but the only macroscopic occupation number is the one of real roots, i.e. only $n_1 \sim N$.

Now we make use of the Destri - de Vega method [27] to derive an integral equation equivalent to the Bethe equations. The derivative of the function

$$A(\lambda) = \log\{1 + (-1)^{n_1} e^{iZ_1(\lambda)}\} \quad (13)$$

acts as a density of real roots and holes in contour integrals; it has first order poles at all allowed $2\pi \times$ (half) integer values of $z_1(\lambda)$, with residue one. Accordingly, for a function f analytic within the contour,

$$\oint_{\Gamma} f(\lambda) A' = \sum_{j:\text{roots}} f(\lambda_j) + \sum_{m:\text{holes}} f(\mu_m). \quad (14)$$

The integration path Γ surrounds the roots and holes counterclockwise. Stretching Γ to $\pm\infty \pm i\epsilon$, one gets the coupled equations for the counting functions of real roots and k -strings

$$z_1(\lambda) = 2N\sigma(\lambda) + 2\text{Im}\left(G_{11} *_{\epsilon} A\right)(\lambda) + \sum_{m=1}^h F_{11}(\lambda - \mu_m) - \sum_{j=1}^{n_k} F_{1k}(\lambda - \zeta_j) + C_1 \quad (15)$$

$$z_k(\lambda) = 2\text{Im}\left(G_{1k} *_{\epsilon} A\right)(\lambda) + \sum_{m=1}^h F_{1k}(\lambda - \mu_m) - \sum_{j=1}^{n_k} F_{kk}(\lambda - \zeta_j) + C_k. \quad (16)$$

These equations generalize the results of Ref. [27] to encompass strings and holes.¹ We have denoted the rapidities of the holes with μ_m , $m = 1, \dots, h$, and the rapidities of the k -strings with ζ_j , $j = 1, \dots, n_k$. The symbol $*_\epsilon$ denotes a convolution along an infinitesimally shifted real line,

$$(G *_\epsilon A)(\lambda) \equiv \int \frac{d\mu}{2\pi} G(\lambda - \mu - i\epsilon) A(\mu + i\epsilon) . \quad (17)$$

The inhomogeneity function σ , arising from the action of an inverse convolution on the momentum function, is

$$\sigma(\lambda) = (\mathbb{1} + \Phi'_{11})^{-1} * p_1 = \arctan \sinh \frac{\pi}{\gamma} \lambda . \quad (18)$$

Similarly, the functions F , giving the hole and string contributions, are defined as

$$F_{1l} = (\mathbb{1} + \Phi'_{11})^{-1} * \Phi_{1l} . \quad (19)$$

The measures of the convolutions have 2π denominators, as in (17), and $\mathbb{1}(\lambda) \equiv 2\pi\delta(\lambda)$. The kernels G of the integral equations are given by differentiation, $G_{1l}(\lambda) = \frac{d}{d\lambda} F_{1l}(\lambda)$, and the effect of the k -string on itself is encoded by the function $F_{kk} = \Phi_{kk} - G_{1k} * \Phi_{1k}$.

It is remarkable that the z_k equation doesn't have any inhomogeneity arising from the momentum functions. This follows from the identity $p_k = G_{1k} * p_1$, which can be proved by Fourier transforming.

Finally, the constant terms in the equations are

$$C_1 = \frac{\pi}{\pi - \gamma} \frac{\theta}{\pi - \gamma} - \frac{\pi(\pi - 2\gamma)}{2(\pi - \gamma)} \left(\left\lfloor \frac{1}{2} - \frac{\theta}{\pi} + S \frac{\gamma}{\pi} \right\rfloor + \left\lfloor \frac{1}{2} - \frac{\theta}{\pi} - S \frac{\gamma}{\pi} \right\rfloor \right) \quad (20)$$

$$C_k = \frac{(2k - 2)\pi}{\pi - \gamma} \frac{\theta}{\pi - \gamma} - \frac{\pi(\pi - k\gamma)}{\pi - \gamma} \left(\left\lfloor \frac{1}{2} - \frac{\theta}{\pi} + S \frac{\gamma}{\pi} \right\rfloor + \left\lfloor \frac{1}{2} - \frac{\theta}{\pi} - S \frac{\gamma}{\pi} \right\rfloor \right) \quad (21)$$

These have the contributions of the twist angle θ , renormalized by the integration over the inverse convolution $(\mathbb{1} + \Phi'_{11})^{-1}$. The integer part terms arise from boundary terms in partial integrations.

For computing the finite size corrections, one takes N large but not infinite. From the functional form of the source term, $2N\sigma$, one can see that the Fermi points, i.e. the limits of the distribution of real rapidities, are close to $\lambda = \pm \log N$. To treat the excitations close to the Fermi points, one defines the so called kink counting functions,

$$z_1^\pm(\lambda) = \lim_{N \rightarrow \infty} \left(z_1(\lambda \pm \frac{\gamma}{\pi} \log 4N) \mp N\pi \right) \quad (22)$$

$$z_k^\pm(\lambda) = \lim_{N \rightarrow \infty} z_k(\lambda \pm \frac{\gamma}{\pi} \log 4N) . \quad (23)$$

The configurations of holes and strings that can be treated exactly in this limit are such that all holes and strings are either close to one of the Fermi points, or well away from both. Here, we shall assume that every hole and string is in the vicinity of one of the Fermi points.

¹The generalization to pure hole excitations was performed in Ref. [16].

We parametrize the hole rapidities as

$$\mu_m = \mu_m^\pm \pm \frac{\gamma}{\pi} \log 4N, \quad m = 1, \dots, h^\pm. \quad (24)$$

The number of holes in the vicinity of the left (right) Fermi point is h^+ (h^-). The total number of holes is $h = h^+ + h^-$. Similarly, the k -string rapidities are $\zeta_j = \zeta_j^\pm \pm \frac{\gamma}{\pi} \log 4N$, and n_k^\pm are the numbers of strings close to each Fermi point. At one Fermi point, the contributions of the holes and strings situated close to the opposite Fermi point are just given by $F(\pm\infty)$.

The only non-trivial scaling behaviour comes from the function σ . Indeed, subtracting the bulk contribution, one has for values of λ close to the Fermi points

$$\lim_{N \rightarrow \infty} \left(2N\sigma(\lambda \pm \frac{\gamma}{\pi} \log 4N) \mp N\pi \right) = \mp e^{\mp \frac{\pi}{\gamma} \lambda}. \quad (25)$$

We get the following coupled non-linear integral equation for the kink counting functions:

$$z_l^\pm(\lambda) = \Omega_l(\lambda) + 2 \operatorname{Im} \left(G_{1l} *_\epsilon A_\pm \right)(\lambda), \quad l = 1, k, \quad (26)$$

where we have used the scaled density integral $A_\pm = \log \left(1 + (-1)^{n_1} e^{iz^\pm} \right)$, and the inhomogeneity functions

$$\Omega_l(\lambda) = \mp \delta_{1,l} e^{-\frac{\pi}{\gamma} \lambda} + \sum_{m=1}^{h^\pm} F_{1l}(\lambda - \mu_m^\pm) - \sum_{j=1}^{n_k^\pm} F_{lk}(\lambda - \zeta_j^\pm) + C_l^\pm. \quad (27)$$

The constant terms carry the renormalized twist θ , as well as the information from the opposite Fermi point:

$$C_l^\pm = C_l + h^\mp F_{1l}(\pm\infty) - n_k^\mp F_{lk}(\pm\infty) \quad (28)$$

For extracting energy and momentum eigenvalues from the scaled NLIEs (26), one needs to know the asymptotic behaviour of the scaled functions. At $\pm\infty$ we have

$$z_1^\pm(\pm\infty) = \pm \pi n_k(k-2) \pm \pi S \mp 2S\gamma + 2\theta. \quad (29)$$

At $\mp\infty$, the kink equation (26) implies

$$z_1^\pm(\mp\infty) = \mp z_1^{\pm'}(\mp\infty) = \mp\infty. \quad (30)$$

Due to the infinitesimal ϵ in $\operatorname{Im} A(\lambda + i\epsilon) \equiv \operatorname{Im}_\epsilon A$, the value of this expression always between $-\pi/2$ and $\pi/2$. In the limit $\epsilon \rightarrow 0$, it is a sawtooth function, with a step-function jump of $-\pi$ at each root or hole position. A small finite ϵ makes the function analytic. The asymptotic values of the imaginary parts of the scaled density integrals are thus

$$\operatorname{Im}_\epsilon A_\pm(\pm\infty) = \operatorname{Im}_\epsilon A(\pm\infty) = \theta \mp S\gamma + \pi \left[\frac{1}{2} - \frac{\theta}{\pi} \pm S \frac{\gamma}{\pi} \right] \equiv \theta_\pm \mp S\gamma, \quad (31)$$

It should be noticed that close to $\pm\infty$ the imaginary part operation can glide through the G_{1l} to act only on A , up to a correction that vanishes like $\epsilon \log \epsilon$. That is, the integrals in the convolutions of Equation (26) are taken along the real axis, so that the kernels G_{1l} are convoluted with the regularized sawtooth function $\operatorname{Im}_\epsilon A$.

In the opposite asymptotic regime, deep in the bulk of filled real rapidity states, the small but finite ϵ lets $z_1^\pm(\mp\infty)$ dominate in the expansion of $A_\pm(\mp\infty + i\epsilon)$, and

$$\operatorname{Im}_\epsilon A_\pm(\mp\infty) = 0. \quad (32)$$

3 The Finite Size Spectrum

At the external rapidity λ close to $i\gamma/2$, the dominating contribution to the eigenvalue of the transfer matrix is given by

$$\Lambda(\lambda; \{\lambda_i^{(k)}\}) = e^{-i\theta} \left[\frac{\sinh(\lambda + i\gamma/2)}{\sinh i\gamma} \right]^{2N} \prod_{l, n_l \neq 0} \prod_{j=1}^{n_l} \frac{\sinh(\lambda_j^{(l)} - \lambda + i(l+1)\gamma/2)}{\sinh(\lambda_j^{(l)} - \lambda - i(l-1)\gamma/2)}. \quad (33)$$

We expect local integrals of motion at the point $\lambda = i\gamma/2$, where the L -matrix (3) degenerates to a permutation operator. The corresponding momentum and energy eigenvalues of a Bethe state are

$$P(\{\lambda_j^{(l)}\}) = i \log \Lambda(i\gamma/2; \{\lambda_j^{(l)}\}) = \left(\pi \sum n_l + \theta + \sum_{l,j} p_l(\lambda_j^{(l)}) \right) \bmod 2 \quad (34)$$

$$E(\{\lambda_j^{(l)}\}) = i \frac{\gamma}{\pi} \frac{d}{d\lambda} \log \Lambda(\lambda; \{\lambda_j^{(l)}\}) \Big|_{\lambda=i\frac{\gamma}{2}} = 2N \frac{\gamma}{\pi} \cot \gamma - \frac{\gamma}{\pi} \sum_{l,j} p'_l(\lambda_j^{(l)}). \quad (35)$$

Using the logarithmic Bethe equations (10), we express the momentum in terms of the quantum numbers I :

$$P(\{\lambda_j^{(l)}\}) = \left(\pi \sum n_l + \frac{\pi}{N} \sum_{l,j} I_j^{(l)} \right) \bmod 2. \quad (36)$$

To compute the energy eigenvalues, we once more employ the contour integral method (14) for the sum over rapidities of real roots $\lambda_j^{(1)}$. Doing this, we get

$$\begin{aligned} E(\{\mu_m\}; \{\zeta_j\}) &= 2N \frac{\gamma}{\pi} \left(\cot \gamma - \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} p'_1(\lambda) \sigma'(\lambda) \right) \\ &+ \frac{\gamma}{\pi} \text{Im}_\epsilon \int_{-\infty}^{\infty} \frac{d\lambda}{\pi} \sigma'(\lambda) A'(\lambda) + \frac{\gamma}{\pi} \sum_m \sigma'(\mu_m) \end{aligned} \quad (37)$$

The dependence of the string rapidities ζ_j is only indirect, coming through the Bethe equations in the NLIE form, when evaluating the last two terms.

The $1/N$ corrections to the energy come from the parts of the second integral in (37) that are close to the Fermi points $\lambda = \pm \frac{\gamma}{\pi} \log 4N$, as well as from the third term. From Equation (18) we have $\sigma'(\lambda) = \frac{\pi}{\gamma} (2 \cosh \frac{\pi}{\gamma} \lambda)^{-1}$, which is exponentially peaked around the origin. From the asymptotic analysis of $\text{Im}_\epsilon A$, we however know that for large N , $\text{Im}_\epsilon A'$ vanishes as a double exponential when one goes in to the bulk from the Fermi points $\pm \frac{\gamma}{\pi} \log 4N$. Thus the main contributions to the second integral in (37) for large N come from the vicinity of these points. In this regime, σ' takes an exponential form, $\sigma'(\mu \pm \frac{\gamma}{\pi} \log 4N) \approx \frac{\pi}{\gamma} \frac{1}{4N} e^{\mp \frac{\pi}{\gamma} \mu}$. Also, the hole rapidities in the last term of (37) are taken to be close to one of the Fermi points, as in Equation (24).

After these considerations, the finite size corrections to the energy are given by

$$E_{\text{FSC}} = \frac{1}{N} (\mathcal{E}_+ + \mathcal{E}_-), \quad (38)$$

where the contributions of the right and left ends of the rapidity spectrum are

$$\mathcal{E}_\pm = \text{Im}_\epsilon \int \frac{d\mu}{\pi} e^{\mp \frac{\pi}{\gamma} \mu} A'_\pm + \sum_{m=1}^{h^\pm} e^{\mp \frac{\pi}{\gamma} \mu_m^\pm}. \quad (39)$$

Using the scaled NLIEs (26), these can further be reduced to

$$\mathcal{E}_\pm = \mp \sum_{m=1}^{h^\pm} z_1^\pm(\mu_m^\pm) \pm \sum_{j=1}^{n_k^\pm} z_k^\pm(\zeta_j^\pm) \pm h^\pm C_1^\pm \mp n_k^\pm C_k^\pm \pm \text{Im}_\epsilon \int \frac{d\mu}{\pi} \Omega_1^{\pm'}(\mu) A_\pm(\mu), \quad (40)$$

where the Ω :s are defined in Equation (27).

The terms in (40) involving the constants C_1 and C_k of Equation (28) can be evaluated using the asymptotic values of the F -functions: $F_{11}(\pm\infty) = \pm\pi \mp \frac{1}{2} \frac{\pi^2}{\pi-\gamma}$ and $F_{1k}(\pm\infty) = \pm k\pi \mp (k-1) \frac{\pi^2}{\pi-\gamma}$.

The values of the counting functions at holes and roots are given by 2π times the (half)integer quantum numbers I^\pm . To evaluate the sums over the hole quantum numbers in (40), we take a distribution where all real roots are in the bulk, so that no holes intermeddle with the roots (Fig 1.). Thus the quantum numbers I_m^+ for the positive rapidity holes get the h^+ largest (half)integer values of $z_1^+(\lambda)/2\pi$, and the quantum numbers I_m^- for the negative holes get the h^- smallest (half)integer values of $z_1^-(\lambda)/2\pi$:

$$I_m^\pm = \pm \frac{1}{2} \left((k-2)n_k + S - 1 \right) - \left[\frac{1}{2} - \frac{\theta}{\pi} \pm S \frac{\gamma}{\pi} \right] \mp 1 \pm m, \quad m = 1, \dots, h^\pm. \quad (41)$$

The quantum numbers $I^{k,\pm}$ of the strings may take any (half)integer values allowed by the asymptotics of the counting functions (11) up to non-monotonicities that are consistent with the non-linear integral equations (26).

Finally, the integral in the last term in (40) can be evaluated by the standard dilogarithm trick, see e.g. [27, 28]. For the densities A_\pm and the inhomogenities Ω^\pm satisfying equations of the form (26), and the boundary conditions (31,32) for A_\pm , we have

$$\text{Im}_\epsilon \int_{-\infty}^{\infty} d\mu \Omega^{\pm'}(\mu) A_\pm(\mu) = \pm \frac{1}{2} \frac{\pi}{\pi-\gamma} (\theta_\pm \mp S\gamma)^2 \mp \frac{\pi^2}{12}. \quad (42)$$

Combining the contributions of the various terms in Eq. (40), the finite size energy of the state described above evaluates to

$$E_{\text{FSC}} = \frac{1}{2N} \left\{ \frac{1}{\pi-\gamma} \left[\theta + \pi\Delta h - \pi(k-1)\Delta n_k \right]^2 + (\pi-\gamma)S^2 - \frac{\pi}{6} \right. \\ \left. + 2\pi \left(k\Delta h \Delta n_k + S n_k(k-1) - n_k^+ n_k^-(1+2k-2k^2) + \sum_{j=1}^{n_k^+} I_j^{(k)+} - \sum_{j=1}^{n_k^-} I_j^{(k)-} \right) \right\} \quad (43)$$

where the antisymmetry of the k -string distribution is $\Delta n_k = n_k^+ - n_k^-$, and the antisymmetry of the hole distribution is

$$\Delta h = \frac{1}{2} \left(h^+ - h^- + \left[\frac{1}{2} - \frac{\theta}{\pi} + S \frac{\gamma}{\pi} \right] + \left[\frac{1}{2} - \frac{\theta}{\pi} - S \frac{\gamma}{\pi} \right] \right). \quad (44)$$

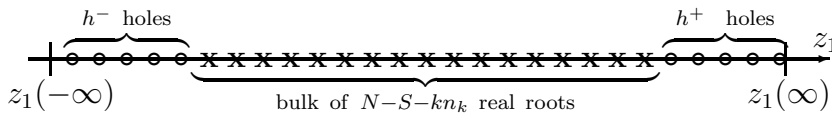


Figure 1: The distribution of real roots and holes in a state with one k -string.

With the distribution of real roots as described, it is an easy task to compute the momentum (36). This turns out to be

$$P = \pi(N - S - kn_k + 1 - \Delta h) \bmod 2\pi + \frac{\pi}{N} \left(kn_k \Delta h + S \left(\frac{\theta}{\pi} + \Delta h \right) + \sum_{j=1}^{n_k} I_j^{(k)} \right) \quad (45)$$

The twisted ground state is the state with no strings nor holes. For zero external magnetic field, it lies in the sector $S = 0$, and has vanishing momentum. The ground state energy is

$$E_{\text{FSC}}^{(0,\theta)} = \frac{1}{2N} \left(\frac{\theta^2}{\pi - \gamma} - \frac{\pi}{6} \right) \quad (46)$$

The central charge is $c = 1$, as the well-known formula of Refs. [2, 3] connecting the ground state energy with the central charge gets corrected in the presence of non-trivial boundary conditions. The central charge is a bulk property, which cannot be changed by boundary effects.

For the subsets of parameter space investigated earlier, the computed excited state energies and momenta agree with results in the literature. For the pure hole states (no complex excitations), they coincide with the ones in Ref. [21]. Regarding states with complex excitations, the two-string result agrees with Ref. [22]. For generic strings, our results agree with the ones of Ref. [18], for the discrete set of couplings ($\gamma = \pi/\nu$, $\nu = 3, 4, \dots$) and twists ($\theta = \gamma$) treated there. When comparing with the literature, it should be noted that our definition of the Hamiltonian by differentiation (35) rescales the Fermi-velocity to 1.

It is also interesting to compare to results obtained by the bosonisation method. From e.g. [30] one can find the excitation spectrum for the untwisted chain. The result for a twisted chain follows from observing that the twist boundary conditions on the chain modify the boundary conditions of the dual boson only. This leads to an energy spectrum which is exactly of the form (43),

$$E_{\text{bosonisation}} = \frac{v_s}{2N} \left(\frac{(m + \theta)^2}{2\pi R^2} + 2\pi R^2 S^2 + 2\pi \sum_{n=1}^{\infty} n(m_n^L + m_n^R) \right). \quad (47)$$

Here $R = \sqrt{\frac{\pi - \gamma}{2\pi}}$ is the compactification radius of the boson, m is the quantized momentum of the boson zero-mode, and $m_n^{L,R}$ are occupation numbers of the n :th left and right moving oscillator modes of the boson. S is the quantized dual boson zero-mode momentum, which coincides with our S . Comparing to (43), we see that Bethe Ansatz strings and holes are mixtures of bosonic zero and oscillatory modes.

4 The Kač table in Liouville theory

As explained in the introduction, imaginary coupling Liouville theory can be described by a collection of twisted XXZ chains, if $\gamma/\pi = \mu/(\nu + 1)$ is rational [17].² The twists are quantized in terms of γ ;

$$\theta = \kappa \gamma, \quad \kappa = 0, \dots, \nu.$$

²More exactly, in [17] the Liouville equivalence was proved only for $\pi/\gamma = \nu + 1$, $\nu = 2, 3, \dots$. The equivalence can be easily extended to more general rational values.

This collection of twisted $\text{XXZ}_{1/2}$ chains is exactly the one that describes the Bethe Ansatz of a root-of-unity $U_q(sl_2)$ invariant periodic chain [13, 14, 15]. This equivalence is not surprising, if one keeps the connection to sine-Gordon theory in mind. The twist-sectors in the lattice Liouville model correspond exactly to the θ -vacua of its massive perturbation, the sine-Gordon model. These vacua become non-degenerate in finite volume (see e.g. [16]). The sine-Gordon model has an $U_q(sl_2)$ symmetry [31, 32], where q is related to the coupling constant $\beta^2 \equiv 8(\pi - \gamma)$.

On the other hand, the integrable lattice discretisation of the sine-Gordon model is an inhomogeneous $\text{XXZ}_{1/2}$ chain with anisotropy γ . The massless limit, i.e. the imaginary coupling Liouville model, is then described by an homogeneous $\text{XXZ}_{1/2}$ chain with $U_q(sl_2)$ symmetry, of the type described in Refs. [13, 14, 15].

The consistent quantum group restriction of the Hilbert space of these root-of-unity $U_q(sl_2)$ invariant models comes about by restricting κ to be nonvanishing. In sine-Gordon theory this leads to the restricted sine-Gordon model, which flows to minimal models in the ultraviolet [31, 32]. Here, we are investigating the Bethe Ansatz description of this ultraviolet limit, and we shall see that the primary operators creating string states furnish the whole Kač table.

The ground state of the quantum group restricted model is in the $\kappa = 1$ sector, reproducing

$$c = 1 - 6\gamma^2/\pi(\pi - \gamma) . \quad (48)$$

For rational γ/π , this gives exactly the central charges of minimal models.

Here it should be noticed that in the quantum group invariant chain, the twist does not arise from boundary conditions on the spatial boundaries of the chain. Rather, different twists select different sectors in the Hilbert space transforming in a specific way under the global \mathbb{Z} -symmetry of the theory. In a bosonized language this symmetry is $\phi \rightarrow \phi + 2n\pi$, $n \in \mathbb{Z}$. This is a symmetry of all theories with the potential a function of $e^{i\phi}$, including the sine-Gordon and imaginary Liouville ones. Thus the situation for a restricted quantum group invariant chain (i.e. a q -restricted Liouville chain) differs from the $\text{XXZ}_{1/2}$ chain with twisted spatial boundary conditions discussed in the previous Section. The equation of Refs. [2, 3], connecting the ground state energy to the central charge, does not get boundary corrections, and the central charge is as above.

Conformal primary and secondary operators create excited states from the vacuum. Following Cardy [4], the critical indices of the operators creating excited states are given by the finite size energies and momenta.

We are interested in primary states with equal holomorphic and anti-holomorphic conformal weights, i.e. with vanishing finite size correction to the momentum (conformal spin). Inspired by Ref. [18], we look for these among the excited states with one higher string ($n_k = 1$). From equation (45) we see that a sufficient condition for the vanishing of the conformal spin is $S = 0$ and

$$k \Delta h + I^{(k)} = 0 . \quad (49)$$

The corresponding energies are

$$E_{\text{FSC}} = \frac{1}{2N} \left\{ \frac{1}{\pi - \gamma} \left[\kappa\gamma - \pi I^{(k)\pm}/k \mp \pi(k-1) \right]^2 - \frac{\pi}{6} \right\} , \quad (50)$$

where the sign refers to the position of the string close to $\lambda = \pm\infty$. The harmonic oscillator pieces in the energies vanish for all of these states.

For generic values of θ and $\gamma < \pi/2$, the k -string may be close to ∞ or $-\infty$, depending on the values of k, γ and θ . Not all string lengths are allowed for all γ and θ , nor all combinations of different hole and string positions. Apart from the Takahashi-Suzuki conditions [33] that restrict allowed string lengths, one has to treat the Bethe Ansatz equation for z_k to find the allowed values of I_k .

For our purposes, it suffices to analyze the asymptotics of $z_k(\zeta)$ for the string position ζ close to $\pm\infty$, which gives the possible range of quantum numbers $I^{(k)}$. We get different branches of primary states solving the condition (49).

The first branch, for which $I^{(k)} = 0$, exists when

$$k + \kappa < \frac{\pi}{\gamma} . \quad (51)$$

For states with hole distributions symmetric up to the twist effects of Equation (49), this k -string is close to $-\infty$.

Using the formula $E_{\text{excited}} - E_o = \frac{\pi}{N}(\Delta + \bar{\Delta})$ of Ref. [4], we get the conformal weights of these primary states:

$$\Delta = \bar{\Delta} = \frac{(\pi(k-1) - \kappa\gamma)^2 - \gamma^2}{4\pi(\pi - \gamma)} . \quad (52)$$

For unitary minimal models ($\gamma = \frac{\pi}{\nu+1}$), these states consisting of one k -string with $I^{(k)} = 0$ and a symmetric distribution of $2k - 2$ holes, furnish the whole Kač table. In the q-restricted case, the remnant κ gets the values $\kappa = 1, \dots, \nu$. The Takahashi-Suzuki condition [33] on the string length allows all strings up to $k = \nu$. For an unitary model, Condition (51) restricts this further to $k < \nu + 1 - \kappa$. Thus the integers κ and k give exactly the two integer labels of the full Kač table, with the appropriate ranges.

For non-unitary minimal models, these states yield only a part of the table. Some primaries are left out due to the technical restriction adopted in this paper, $k\gamma < \pi$.

5 Conclusions

We have computed the finite size corrections to the energies and momenta of excited states in $\text{XXZ}_{1/2}$ chains periodic up to a twist, for generic anisotropy $\gamma < \pi/2$ and twist θ . The excited states were described as holes in the sea of real Bethe Ansatz rapidities, as well as complex rapidities collected in (positive parity) strings. We restricted ourselves to the case where there are strings of only one specified length k in addition to the real rapidities.

For extracting the scaling information, we used the Destri-de Vega approach to treat Bethe Ansatz equations, generalizing it to cope with string-like excited states.

Apart from completing the picture of excitation energies to be found from the literature, to arbitrary coupling, twist and string length (up to $k < \pi/\gamma$), our result (43,45) is interesting due to its special dependence of the “field theoretic” quantum numbers h^\pm . This we consider the major novel feature of our finite size solution.

One of the original motivations to develop the DdV formalism was the possibility to derive thermodynamic Bethe Ansatz (TBA) equations directly from the algebraic

Bethe Ansatz [27]. The field theoretic TBA degrees of freedom correspond to holes and strings, which are excitations above the vacuum of interacting magnons. The holes correspond to right and left moving kinks, and the strings correspond in this case to weakly bound kinks, with binding energy of the order $\sim 1/N$.

As can be seen from Equation (44), the twist dependence of the energy for a fixed field-theoretical configuration is not innocent. Due to the integer part cuts, the energies and momenta have discontinuities at some specific values of the twist. From the point of view of the original magnon degrees of freedom $\lambda_j^{(l)}$ of the Bethe Ansatz equations (10), however, an adiabatic excursion in θ is possible. The $\lambda_j^{(l)}$:s are the building blocks of the underlying interacting vacuum, and the energies and momenta are continuous if the $\lambda_j^{(l)}$:s move continuously.

During such an adiabatic excursion, the number of left and/or right holes change when θ moves over a value giving a jump in the integer part expressions of (44). Thus at these values of θ the vacuum absorbs a left kink state and emits a right one, or vice versa. It should be stressed that this spectral flow is not just a trivial consequence of the choice of branch in the counting functions. Rather, it encodes drastic changes in the structure of the Hilbert space for different values of θ . For example, the degeneracies of secondary states built by driving some of the holes into the bulk change at the jump points.

Adiabatic excursions of this type have been investigated in the context of a XXZ chain threaded by a magnetic flux [23, 24, 25]. For this, the twist is given the physical interpretation as a threading magnetic flux. For full comparison with the numeric excited state results of [24, 25], negative parity strings have to be incorporated into our treatment, as these play a central role for larger values of γ .

Finally, we showed how in the q -restricted Hilbert space of the $U_q(sl_2)$ invariant periodic chain, which is equivalent to the imaginary coupling Liouville model and the UV-limit of sine-Gordon, the central charge decreases from $c = 1$ to those of minimal models. Excited states with one higher string and a minimal amount of holes, give rise to all the primary states in the case of unitary minimal models.

To conclude, we want to comment that the string picture should not be necessary for deriving the results of this paper. In Ref. [28], wide and close pairs were used to describe complex excitations. This approach could certainly be generalized to twisted boundary conditions in the XXZ chain. The strings, however, provide a good tool which singles out very specific combinations of wide and close pairs that correspond to primary states in the critical theory.

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